Low-Energy Equivalent-Potential Approach for Strong Interactions*

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Equations are given for constructing an equivalent potential at each energy from the crossed-channel absorptive part. These are straightforward generalizations of the Charap-Fubini formulas. The absorptive part can be calculated with the help of the strip approximation. If we solve our equations by iteration, we can get the potential more and more accurately at smaller distances. The first approximation consists of an appropriate number of crossed-channel partial waves which can be calculated by solving a Schrödinger equation with an equivalent potential as input. The approach is applied in lowest order to calculate the parameters of the ρ and φ mesons.

1. INTRODUCTION

A WAY of defining a potential in strong interactions was given by Charap and Fubini.[1] The potential was chosen so that solving the Schrödinger equation would give the correct relativistic amplitude at zero kinetic energy. This potential was shown to have a wide range of validity. An explicit iterative scheme was also proposed for constructing such a potential from the crossed-channel absorptive part.

In Sec. 2 we shall generalize the Charap-Fubini approach by requiring that our potential reproduce the relativistic scattering amplitude at any given energy.[2] Such an equivalent potential will, of course, be different at different energies. The absorptive part which we need can be built up through an iterative solution of the strip-approximation equations (Sec. 3). The lowest approximation will simply consist of an appropriate number of crossed-channel partial waves. These can, in turn, be calculated by solving a Schrödinger equation with an equivalent potential as input. Although such an equation does not have, perhaps, any physical significance except in the nonrelativistic region, it does serve to enforce unitarity. This is just what the N/D method does in the more usual approach.

The procedure we just described should give a reasonable long-range force in lowest order. As we go to higher orders we should get an increasingly accurate description of short-range effects. Thus the method is essentially a low-energy method, since long-range forces are more important at low energies. This would be particularly true if we had repulsive effects at small distances. As we go to higher energies we would have to treat short-range forces more and more accurately. We would also have to go beyond two-body scattering.

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[1] G. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959); **15**, 73 (1960).

[2] A similar approach has been used by A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963), and by A. A. Logunov, A. N. Tavkhelidze, I. T. Todorov, and O. A. Khrustalev, Nuovo Cimento **30**, 134 (1963). However, they restricted themselves to perturbation theory and did not use the Schrödinger equation with a local potential.

2. CONSTRUCTION OF THE EQUIVALENT POTENTIAL

Consider the Schrödinger equation for two particles

$$\nabla^2\psi+[k^2-V(r,q^2)]\psi=0, \tag{1}$$

where ψ = wave function, k = magnitude of the three-momentum in the c.m. system, r = radial distance, and $V(r,q^2)$ is the potential which gives the correct relativistic amplitude when $k^2=q^2$. For simplicity we shall only look at the spinless one-channel problem for particles of unit mass, although it would be straightforward to extend our results to a more general case. We have also ignored exchange forces, since their inclusion does not change our general arguments. If our potential has the form

$$V(r,q^2)=-\pi^{-1}\int_{t_0}^{\infty}dt'v(t',q^2)r^{-1}e^{-r\sqrt{t'}}, \tag{2}$$

the physical scattering amplitude $f(k^2,t)$ satisfies the Mandelstam representation.[3] Here f is such that the differential cross-section $=|f|^2$, while $t=-2k^2(1-\cos\theta)$ and θ = scattering angle in the c.m. system. Thus we may write

$$f(k^2,t)=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{f_t(t',k^2)}{t'-t}, \tag{3}$$

where f_t would be the absorptive part in the t channel, if there were a t channel. It can be written in terms of the double-spectral function $\alpha(k^2,t)$ as

$$f_t(t,k^2)=v(t,q^2)+\frac{1}{\pi}\int_0^{\infty}dk'^2\frac{\alpha(k'^2,t)}{k'^2-k^2}. \tag{4}$$

From unitarity we find in the usual way[4]

$$\alpha(k^2,t)=\frac{1}{2\pi k}\int_{t_0}^{\infty}dt'\int_{t_0}^{\infty}dt''\frac{f_t^*(t',k^2)f_t(t'',k^2)}{K^{1/2}(k^2;t,t',t'')}\times\theta[t-t_+(k^2)], \tag{5}$$

where

$$K(k^2;t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'')-tt't''k^{-2} \tag{6}$$

[3] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

[4] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

and

$$t_+(k^2)=t'+t''+\frac{t't''}{k^2}+2\left[(t't'')\left(1+\frac{t'}{4k^2}\right)\left(1+\frac{t''}{4k^2}\right)\right]^{1/2}. \quad (7)$$

Consider now the Lorentz-invariant amplitude $A(s,t)$ when the c.m. three-momentum is q. This is related to the physical amplitude f through $A=(q^2+1)^{1/2}f$, where $s=4(q^2+1)=$square of the total c.m. energy. If we assume that A satisfies the Mandelstam representation we have

$$A(s,t)=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{A_t(t',s)}{t'-t}, \quad (8)$$

where A_t is the absorptive part in the t channel and t_0 is the square of the mass of the lowest intermediate state in that channel. Comparing Eqs. (3) and (8) we see that the requirement that the Schrödinger equation give the correct relativistic amplitude at $k^2=q^2$ is equivalent to putting

$$f_t(t,q^2)=2s^{-1/2}A_t(t,s). \quad (9)$$

If we now subtract from Eq. (4) its value at $k^2=q^2$ and use Eq. (9), we have

$$f_t(t,k^2)=2s^{-1/2}A_t(t,s)+\frac{k^2-q^2}{\pi}\times\int_0^{\infty}dk'^2\frac{\alpha(k'^2,t)}{(k'^2-q^2)(k'^2-k^2)}, \quad (10)$$

which together with Eq. (5) forms a nonlinear integral equation for $f_t(t,k^2)$ for any given $A_t(t,s)$. This equation can be solved by iteration. Suppose we associate a parameter λ with A_t and drop the last term in Eq. (10). Then because of the θ function in Eq. (5) the resulting f_t will be correct in the region $t_0<t<4t_0$. If we then insert this f_t into Eq. (5), the resulting Eq. (10) will give f_t correctly in the region $t_0<t<9t_0$. In general, if we keep only terms up to order λ^n, we will have f_t exact in the region $t_0<t<(n+1)^2t_0$.

Once we know f_t and α from Eqs. (5) and (10), we can find the potential by combining Eqs. (9) and (4). This gives

$$v(t,q^2)=2s^{-1/2}A_t(t,s)-\frac{1}{\pi}\int_0^{\infty}dk'^2\frac{\alpha(k'^2,t)}{k'^2-q^2}. \quad (11)$$

Thus, if we use the above iteration scheme and keep only terms up to order λ^n, Eq. (11) will give $v(t,q^2)$ correctly in the region $t_0<t<(n+1)^2t_0$. Of course, we have been assuming that we know A_t exactly. If we go only up to order λ^n we would need it only for $t<(n+1)^2t_0$. Once we know the potential we can solve the equation

$$\nabla^2\psi+[q^2-V(r,q^2)]\psi=0 \quad (12)$$

to obtain the correct amplitude at the energy corresponding to q^2. Since the above procedure can be carried out for any value of q^2, we can get the correct amplitude at any energy.

The above method is particularly useful if A_t has to be known accurately only for small values of t. Of course, it never has to be known in the region where the Regge behavior[5] dominates for both A_t and α, since then the two terms in Eq. (11) can be shown to cancel.[6] However, it would not even have to be known too accurately for moderate values of t if v should turn out to be either weak or repulsive there. The latter would naturally be the most desirable case. While nothing can really be said until an actual calculation is carried out, Eq. (11) suggests that this may not be an entirely farfetched possibility, since the last term in that expression may well turn out to correspond to a repulsion, at least for small values of q^2. If it also overwhelms the first term at moderate t we would in fact have a potential with a repulsive core.

3. THE CROSSED-CHANNEL ABSORPTIVE PART

One way of obtaining $A_t(t,s)$ is through the strip approximation.[7] We shall only consider a somewhat simplified version of this method. In lowest order we assume that A_t can be expressed in terms of a small number of partial waves in the t channel. It thus has the form

$$A_t^{(0)}(t,s)=\sum_{l=0}^{L}(2l+1)\,\mathrm{Im}A_l(t)P_l[1+(2s/t-4)], \quad (13)$$

where A_l is a partial-wave amplitude. Since A has been assumed to satisfy a Mandelstam representation, we have

$$A_t(t,s)=A_t^{(0)}(t,s)+\frac{1}{\pi}\int_4^{\infty}ds'\rho(s',t)\left[\frac{1}{s'-s}-\frac{2}{t-4}\times\sum_{l=0}^{L}(2l+1)Q_l\left(1+\frac{2s'}{t-4}\right)P_l\left(1+\frac{2s}{t-4}\right)\right], \quad (14)$$

where we have subtracted out the lowest order term. This subtraction has the effect of suppressing the high s' part of the integral in Eq. (14), and so we need consider only that part of the double-spectral function ρ which comes from elastic s-channel unitarity.

From unitarity we have[4]

$$\rho(s,t)=\frac{1}{\pi q\sqrt{s}}\int_{t_0}^{\infty}dt'\int_{t_0}^{\infty}dt''\frac{A_t^*(t',s)A_t(t'',s)}{K^{1/2}(q^2;t,t',t'')}\times\theta[t-t_+(q^2)], \quad (15)$$

where K and t_+ are defined by Eqs. (6) and (7). As in the preceding section, we can associate the parameter λ with $A_t^{(0)}$ and use Eqs. (14) and (15) to set up an iteration scheme. If we drop the integral term in Eq. (14) we will have A_t for $t_0<t<4t_0$ because of the θ function in Eq. (15). If we insert this lowest order term into Eq. (15), Eq. (14) will give A_t for $t_0<t<9t_0$; in

[5] T. Regge, Nuovo Cimento **14**, 951 (1959).

[6] The author would like to thank Professor B. M. Udgaonkar and U. Trivedi for a discussion on this point.

[7] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

general, Eq. (14) will then diverge, so we may have to subtract out an additional number of waves to guarantee convergence. If we repeat this procedure until we have all the terms up to order λ^n, we will have A_t in the region $t_0<t<(n+1)^2t_0$.

The above scheme presupposes a knowledge of the low partial waves. However, these can be found by solving the Schrödinger equation (12) in the crossed channel. In particular, this will give the bound states and resonances. Of course, our method for obtaining A_t is valid only for moderate energies, since otherwise Eq. (13) will no longer be meaningful.

Finally, it should be noted that the procedure of this section leads to real v. Suppose we keep only terms up to order λ^n in the iteration schemes of both Secs. 2 and 3. This will give v correctly in the region $t_0<t<(n+1)^2t_0$. Now Eqs. (11) and (14) give

$$\mathrm{Im}v(t,q^2)=2s^{-1/2}\rho(s,t)-\alpha(q^2,t).$$

But we are requiring Eq. (9) in each order. Thus Eqs. (5) and (15) will give $2s^{-1/2}\rho(s,t)=\alpha(q^2,t)$, if we keep all the terms up to order λ^n in each case. Hence, $\mathrm{Im}v=0$.

4. LOWEST ORDER CALCULATION OF THE ϱ AND φ MESONS

We shall now illustrate the above method by doing a crude lowest order calculation of the ρ. Assuming that the ρ meson dominates A_t in the $\pi\pi$ problem, we have in the s-channel $I=1$ state

$$A_t(t,s)=3\beta_{11}\,\mathrm{Im}A_1(t)P_1[1+(2s/t-4)], \quad (16)$$

where the crossing matrix element connecting $I=1$ in the t channel to $I=1$ in the s channel is $\beta_{11}=\frac{1}{2}$. We shall, as usual, make a delta-function approximation for the resonance

$$\mathrm{Im}A_1(t)=4\pi\Gamma_1 q_R^2\delta(t-m^2), \quad (17)$$

where $2(q_R^3\Gamma_1/m)$ is the half-width in the q^2 variable, $q_R^2=\frac{1}{4}m^2-1$, and m is the mass of the ρ. Combining Eqs. (16) and (17) we see from Eqs. (11) and (2) that in lowest order the effective potential in the $I=1$, $l=1$ state is

$$V(r,q^2)=-24\beta_{11}\Gamma_1 s^{-1/2}(s+2q_R^2)r^{-1}e^{-mr}, \quad (18)$$

where the extra factor of 2 comes from the fact that we have contributions from both t and u channels.

To solve Eq. (12) we shall use a standard expression[8] for the phase shift

$$\cot\delta_1=\left[\int_0^\infty dr\,qrj_1(qr)Vu_1\right]^{-2}\left[q\int_0^\infty dr\,Vu_1^2 -2\int_0^\infty dr\,qrn_1(qr)V(r,q^2)u_1(r)\times\int_0^r dr'qr'j_1(qr')V(r',q^2)u_1(r')\right], \quad (19)$$

which is stable with respect to variations in the wave-function u_1. Thus we should get a reasonable result even with a crude approximation for u_1 which, moreover, does not have to be normalized in any way or be correct outside the range of the force. We therefore take roughly the form we can expect it to have if the ρ were a bound state

$$u_1(r)=r^2e^{-mr}, \quad (20)$$

and assume it to be energy-independent. Finally, we make an effective-range approximation by expanding about $q^2=0$ to obtain

$$2s^{-1/2}q^3\cot\delta_1=a_e-r_eq^2, \quad (21)$$

where

$$a_e=m^3\left[\frac{128m}{81\Gamma_1(m^2+4)}-\frac{1}{2}\right] \quad (22)$$

and

$$r_e=\frac{m}{32}(37-8m^2)-\frac{128m^2}{81\Gamma_1(m^2+4)}\left[1-\frac{8m^2}{m^2+4}\right]. \quad (23)$$

Equation (21) will give a resonance at $q^2=a_er_e^{-1}$, with a reduced width $\Gamma_1=r_e^{-1}$. The latter relation can be trivially solved to find Γ_1 in terms of m. Using this result we can then try various values of m and see whether the condition $q_R^2=a_er_e^{-1}$ is satisfied. Approximate self-consistency was achieved with $\Gamma_1=0.55$ and $m=3.9$. This is to be compared with the experimental values of $\Gamma_1=0.16$ and $m=5.5$ (765 MeV). A plot of the partial-wave cross section obtained from Eq. (21) gave a width of about 160 MeV. The corresponding experimental value is 100 MeV.

The above calculation can also be carried out for the φ meson in $K\bar{K}$ scattering. We assume that ρ exchange dominates and determine the $\rho K\bar{K}$ coupling constant in terms of the experimental $\rho\pi\pi$ coupling through $SU(3)$. If we take the experimental ρ mass, the potential is fixed and there is no self-consistency problem. Equation (21) now gives a resonance in the $I=0$, $l=1$ state, with mass $=1435$ MeV and reduced width $=0.021$. The corresponding experimental values are 1020 MeV and 0.010, respectively, if the full width is taken to be 3 MeV.

5. ALTERNATIVE TECHNIQUE FOR CONSTRUCTING THE EQUIVALENT POTENTIAL

An alternative way of constructing the equivalent potential is to use the Lippmann-Schwinger equation corresponding to (1)

$$\Phi(\mathbf{k}',\mathbf{k})=W[-(\mathbf{k}'-\mathbf{k})^2,q^2]+\int\frac{d^3p}{(2\pi)^3}\frac{W[-(\mathbf{k}'-\mathbf{p})^2,q^2]\Phi(\mathbf{p},\mathbf{k})}{p^2-k^2-i\epsilon}, \quad (24)$$

where W is the Born approximation of the potential

$$W(t,q^2)=\frac{1}{\pi}\int_{t_0}^\infty dt'\frac{v(t',q^2)}{t'-t}, \quad (25)$$

[8] See, for instance, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1128.

and Φ is such that, when $k'^2=k^2$, we have $\Phi(\mathbf{k}',\mathbf{k})=f(k^2,t)$. Fubini[9] has shown that we can write

$$\Phi(\mathbf{k}',\mathbf{k})=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{\varphi(k'^2,k^2,t')}{t'+(\mathbf{k}'-\mathbf{k})^2}, \tag{26}$$

with φ satisfying the equation

$$\varphi(k'^2,k^2,t)=v(t,q^2)+\int_0^{\infty}\frac{dp^2}{8\pi^2}\int_{t_0}^{\infty}dt' \times\int_{t_0}^{\infty}dt''R(k'^2,t;p^2,t';t'',k^2) \times\frac{v(t'',q^2)\varphi(p^2,k^2,t')}{p^2-k^2-i\epsilon}, \tag{27}$$

where

$$R=\theta(\sqrt{t}-\sqrt{t'}-\sqrt{t''})\theta(\Delta)/\sqrt{\Delta} \tag{28}$$

and

$$\Delta=-\tfrac{1}{2}\begin{vmatrix} 2k'^2 & (t''+k'^2+p^2) & (t+k^2+k'^2) \\ (t''+k'^2+p^2) & 2p^2 & (t'+k^2+p^2) \\ (t+k^2+k'^2) & (t'+k^2+p^2) & 2k^2 \end{vmatrix}.$$

From Eqs. (8) and (26) we see that the requirement that the solution of Eq. (27) give the correct relativistic amplitude at $k^2=q^2$ is equivalent to putting

$$\varphi(q^2,q^2,t)=2s^{-1/2}A_t(t,s). \tag{29}$$

Combining this with Eq. (27), we have

$$v(t,q^2)=2s^{-1/2}A_t(t,s)-\int_0^{\infty}\frac{dp^2}{8\pi^2}\int_{t_0}^{\infty}dt' \times\int_{t_0}^{\infty}dt''R(q^2,t;p^2,t';t'',q^2) \times v(t'',q^2)\varphi(p^2,q^2,t')/(p^2-q^2-i\epsilon). \tag{30}$$

To find φ it is convenient to substitute this into Eq. (27) for $k^2=q^2$

$$\varphi(k'^2,q^2,t)=2s^{-1/2}A_t(t,s)+\int_0^{\infty}\frac{dp^2}{8\pi^2}\int_{t_0}^{\infty}dt' \times\int_{t_0}^{\infty}dt''[R(k'^2,t;p^2,t';t'',q^2) -R(q^2,t;p^2,t';t'',q^2)] \times v(t,''q^2)\varphi(p^2,q^2,t')/(p^2-q^2-i\epsilon). \tag{31}$$

For a given $A_t(t,s)$, Eqs. (30) and (31) constitute a pair of simultaneous integral equations for φ and v. As in Sec. 2, they can be solved by iteration. In lowest order we just drop the last terms in Eqs. (30) and (31). Then because of the θ function in the definition (28) of R, the resulting v and φ will be exact in the region $t_0<t<4t_0$. If we insert these into the integrals of Eqs. (30) and (31) we will have v and φ correct for $t_0<t<9t_0$, etc.

The above procedure is probably more complicated than the one in Sec. 2 since we have to solve two equations simultaneously. Of course, the two methods are completely equivalent. This is true whether we solve them exactly, or expand in the parameter λ which we associated with A_t. The latter procedure, as we saw, leads to a v correct in the region $t_0<t<(n+1)^2t_0$ if we keep only terms up to order λ^n.

One advantage of the method of this section is that it does not depend on the validity of the full Mandelstam representation for potential scattering or on elastic unitarity. Thus we could construct an equivalent potential even if we had inelastic effects; such a potential would in general be complex. Suppose, for instance, that we have some grounds for believing that A_t for low t is well approximated at some energy by a few t-channel Regge poles, but that the full amplitude in the physical region is not. Then if only the long-range part of the potential is important, or if some model of the short-range part can be found, we should be able to get a reasonable physical amplitude by solving Eq. (12).

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[9] S. P. Fubini, *Theoretical Physics, Lectures Presented at a Seminar, Trieste, 16 July–25 August 1962* (International Atomic Energy Agency, Vienna, 1963), p. 347.